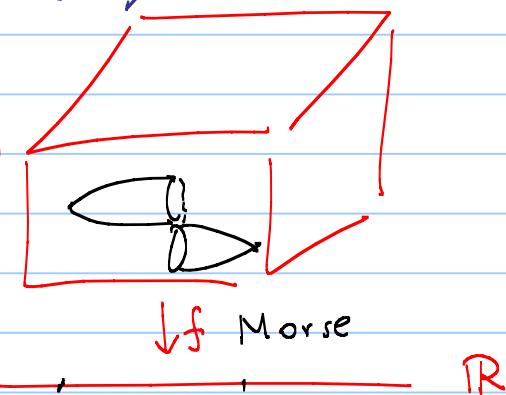


# Haydys, Fukaya-Seidel Category & Gauge theory.

Morse theory

- Classical: study  $H_*(M)$  in term of data on a regular fiber.



- Witten approach

Define  $H_*(M)$  in terms of flow lines of  $f$

Aim: A construction in the spirit of Witten's idea  $\overbrace{\text{Morse theory}}$ .

Symplectic Lefschetz fibration

$(M, \omega, J)$  sympl. mfd.

$f: M \rightarrow \mathbb{C}$   $J$ -holo.

- $\omega = d\lambda$  exact

- $m_1, \dots, m_k$  non-degenerated crit. pt.  
 $z_1, \dots, z_k$  critical values (distinct)

$z_0 \in \mathbb{C} \setminus \{z_1, \dots, z_k\}$

$M_0 = f^{-1}(z_0)$  reg. fiber

Seidel '02: Associates an inv. to  $f$

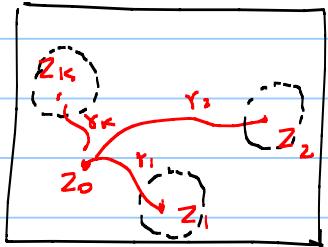
$f \xrightarrow{(1)}$  Collection  $\Gamma_f$  of  $k$  Lagr. spheres in  $M_0$

$\xrightarrow{(2)}$  Category  $\overrightarrow{\text{Lag}}(\Gamma_f)$  (depend on choices)

$\xrightarrow{(3)}$   $D^b(\overrightarrow{\text{Lag}}(\Gamma_f))$  (indep. of choices)

Step 1. Fix  $m_j$   
 $\exists$  nbd.  $U$  of  $m_j$   
s.t.  $f^{-1}(z) \cap U \cong T^*S_z$   $\xrightarrow{\text{symp.}}$

provided  $z \in B_\varepsilon(z_j) \setminus \{z_j\}$



$L_j$ : parallel transport  
of the vanishing cycle  
of  $m_j$  along  $\gamma_j$ .

$L_j \subset M_0$  Lagr. sphere

$$\Gamma_f := (L_1, \dots, L_k)$$

Step 2. Assume  $L_i \pitchfork L_j$ 's

$$Ob(\vec{\text{Lag}}(\Gamma_f)) = \{L_1, \dots, L_k\}$$

$$\text{hom}(L_p, L_q) = \begin{cases} \langle L_p \cap L_q \rangle & p > q \\ \mathbb{Z}/2\mathbb{Z} & p = q \\ 0 & p < q \end{cases}$$

~~Diagram showing a sequence of vertical lines with a red oval around them, labeled J-holo. An arrow points from this to the homomorphism formula above.~~

Step 3  $FS(f) = D^b(\vec{\text{Lag}}(f))$

Theorem (Seidel '02)  $FS(f)$  inv. of  $f$



Alternative construction:

$f \xrightarrow{(a)} \text{Category } A_f$  (dep. on choices)  
 $\xrightarrow{(b)} D^b(A_f)$  (indep. of choices)

$\text{Ob}(A_f) :$   $m_1, \dots, m_k$

$$\text{hom}(m_p, m_q) = \begin{cases} \mathbb{C}F(m_p, m_q) & p > q \\ \mathbb{Z}/2\mathbb{Z} & = \\ 0 & < \end{cases}$$

$\gamma : \mathbb{R} \rightarrow M$

$$\lim_{t \rightarrow +\infty} \gamma(t) = m_p$$

$$\lim_{t \rightarrow -\infty} \gamma(t) = m_q$$

$$J(\gamma) := \int_{\mathbb{R}} \gamma^*(\lambda) - \int_{\mathbb{R}} \text{Im}(e^{i\theta(t)} f(\gamma(t)) dt$$

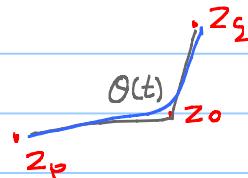
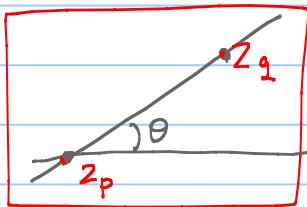
Symp. Action fcl.

$\omega = d\lambda$        $\theta(t)$  'suitable' choice.

$$\text{Crit}(J) : \left\{ \dot{\gamma} + \text{grad Re}(e^{i\theta} f) \cdot \gamma = 0 \right. \quad (*)$$

w/ bdy condition

better choice  $\theta(t)$ .

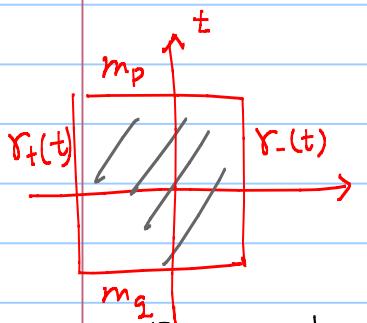


$\mathbb{C}F(m_p, m_q) \triangleq$  Morse-Witten cpx. of  $J$

generators: sol<sup>1/2</sup> of  $(*)$

$\partial$ : for  $\gamma_{\pm}$  (sol<sup>1/2</sup> of  $(*)$ )

$u : \mathbb{R}^2 \rightarrow M$

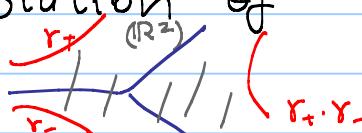


$$\partial_s u + J(\partial_t u + \text{grad Re}(e^{i\theta} f) \cdot u) = 0$$

$$\lim_{t \rightarrow \infty} u(s, t) = m_p \quad \lim_{t \rightarrow -\infty} u = m_q$$

$$\lim_{s \rightarrow \pm\infty} u(s, t) = \gamma_{\pm}(t) \quad (\Rightarrow E(u) < \infty)$$

Remark 1: Composition in  $A_f$  is given by counting "solution of  $(**)$  on a pair-of-pants."



Remark 2:  $H_1(\mathbb{C}F(m_p, m_q))$  is generated by flow lines

< Main results >

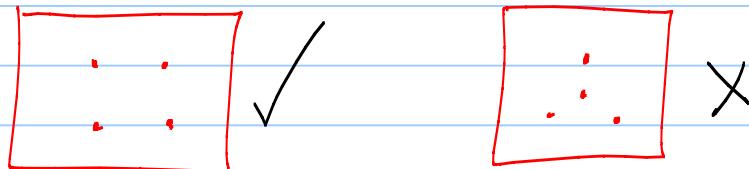
Theorem: Problem  $(\star\star)$  is Fredholm.

(elliptic  $\checkmark$ , but non-cpt. (not cylinder))

$$\hat{\mathcal{M}}(r_-, r_+) \triangleq \{ u \text{ solves } (\star\star) \} / \mathbb{R}$$

$$\hat{\mathcal{M}}(m_p, m_q) \triangleq " \bigcup_{r_+, r_-} \hat{\mathcal{M}}(r_-, r_+) " \quad m_p \circlearrowleft \begin{array}{c} \diagup \diagdown \\ \diagup \diagdown \end{array} m_q \quad (\exists \text{ breaking})$$

Theorem 2: Assume critical values  $z_1, \dots, z_k$  are in convex position



and fibration  $f: M \rightarrow \mathbb{C}$  is trivial at  $\infty$   
 $\Rightarrow \hat{\mathcal{M}}(m_p, m_q)$  is compact.

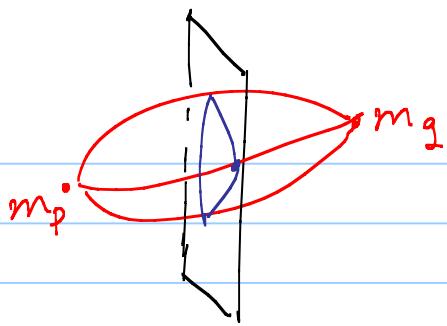
Main steps of the pf.:

Prop. (Energy identity)  $\forall u \in \mathcal{M}(r_-, r_+)$

$$E(u) := \int_{\mathbb{R}^2} |\partial_s u|^2 ds dt + F(r_-) - F(r_+) < \infty$$

Prop.  $\exists$  a priori  $C^0$  bound on  $u$   
 $\Rightarrow C^1$ -bound ( $\& C^\infty$ -bound).

Prop. The energy of  $u$  does not leak as  $t \rightarrow \pm\infty$



Gauge theory

$$P : \text{PSL}_2(\mathbb{C}) - \text{bdy} / Y^3$$

$$CS : A(P) \longrightarrow \mathbb{C}$$

$$\text{Crit}(CS) = \text{flat PSL}_2(\mathbb{C})\text{-conn}$$

grad. flow lines of  $\text{Re}(e^{i\theta} f)$   $\rightarrow$  Vafa-Witten eg. on  $Y \times \mathbb{R}$   
 Kapustin-Witten eg. on  $Y \times \mathbb{R}$

pseudo-holo. plane  $\rightarrow$  Haydys-Witten eg. on  $Y \times \mathbb{R}$

- Calabi-Yau 3-fold  $Y^3 \mathbb{C}$   
 $E \rightarrow Y$  cx. VB

$\text{Crit}(CS) = \text{holo. str. on } E$

flow lines of  $\text{Re}(e^{i\theta} CS)$  :  $G_2$ -instanton on  $Y \times \mathbb{R}$

pseudo-holo. planes :  $\text{Spin}(7)$ -instantons on  $Y \times \mathbb{R}^2$